

Stationary generalized Kerr–Schild spacetimes

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Abstract

In this paper we have applied the generalized Kerr–Schild transformation finding a new family of stationary perfect–fluid solutions of the Einstein field equations. The procedure used combines some well–known techniques of null and timelike vector fields, from which some properties of the solutions are studied in a coordinate–free way. These spacetimes are algebraically special being their Petrov types II and D. This family includes all the classical vacuum Kerr–Schild spacetimes, excepting the plane–fronted gravitational waves, and some other interesting solutions as, for instance, the Kerr metric in the background of the Einstein Universe. However, the family is much more general and depends on an arbitrary function of one variable.

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I Introduction

Since the discovery of the Kerr metric [1] until now, the Kerr–Schild Ansatz [2]–[4] has been a powerful tool to find new explicit solutions of the Einstein equations. Roughly speaking, this technique consists in generating new solutions from the Minkowski spacetime and its null geodesic vector fields, and it can be applied to several types of energy–momentum content: vacuum, electromagnetic field, pure radiation, etc. With regard to the results obtained, we should note that some of the solutions found have been of crucial importance in general relativity and they have received too much attention, specially in relation to the description of black holes or radiating bodies. Well known examples are the Kerr–Newman family of spacetimes, the Vaidya metric, etc. A complete review of the classical Kerr–Schild transformation can be found in reference [5] (chapter 28).

Some generalizations of this generation technique have appeared, so that they overcome some of the limitations that the classical Kerr–Schild transformation had. In particular, the energy–momentum tensor of the classical Kerr–Schild spacetimes generated have the null vector field used in the transformation as a null eigenvector and hence, they cannot be perfect–fluid solutions. A generalization which allows to find perfect–fluid spacetimes is the generalized Kerr–Schild (GKS hereafter) transformation, which has been studied widely in [6]–[15]. The GKS transformation is a generation technique in which the metrics of the initial and the final spacetimes are related by the following expression

$$\tilde{g}_{ab} = g_{ab} + 2\mathcal{H}\ell_a\ell_b, \quad (1)$$

where the starting metric g_{ab} , which we shall call *seed metric*, is any metric, $\vec{\ell}$ is a null vector field for the metric g_{ab} , and \mathcal{H} is a scalar field. In the case in which $\vec{\ell}$ is geodesic, an illuminating result was proved in [6, 7] (see also [8, 9, 10]). It states that the energy–momentum tensor of the GKS spacetimes has $\vec{\ell}$ as an eigenvector if and only if $\vec{\ell}$ is an eigenvector for the energy–momentum tensor of the seed spacetimes. Therefore, if we are interested in finding perfect–fluid spacetimes, we should start with a seed spacetime whose energy–momentum tensor does not have null eigenvectors. In this line, a particularization of the GKS transformation for the search of perfect–fluid solutions was developed in [6, 7], where the seed spacetime was taking to be conformally–flat. From the application of this technique to several cases, new perfect–fluid solutions of Einstein equations have been obtained. In some of these applications the solutions represent inhomogeneous cosmological models while in others, they are static and stationary spacetimes (see [6]–[9], [12] and [16]).

The subject of this paper is to apply the GKS transformation, in combination with techniques of null and timelike vector fields, in order to find new perfect–fluid solutions of Einstein’s equations. The result found is a wide new family of stationary perfect–fluid spacetimes.

In the development of this work we follow the spirit of the classical Kerr–Schild transformation, where the seed spacetime was very simple (the Minkowski spacetime) and the richness of the transformation lay on the choice of the null vector fields $\vec{\ell}$ used. In this sense, we apply the GKS transformation to the FLRW models with constant scale factor and taking their most general shear-free geodesic null (SFGN hereafter) vector field as the vector field $\vec{\ell}$. These objects, which are the basic ingredients to carry out the GKS transformation, are studied in section II. In that situation, we study the perfect-fluid Einstein field equations for the GKS metrics, from which we obtain the form of the GKS energy-momentum tensor (energy density, pressure and fluid velocity) and some partial differential equations for the function \mathcal{H} (section III). Then, through the study of the kinematical quantities of the GKS fluid velocity we impose some restrictions in order to find stationary perfect-fluid solutions. Moreover, these restrictions together with the choice of the null basis will allow us to integrate the equations for \mathcal{H} without the explicit use of a coordinate system. In addition, we deduce the Petrov type of the GKS spacetimes as well as expressions for the non-zero kinematical quantities of the fluid velocity (section IV).

In section V, we find the explicit form of the solutions. As far as we know, this family was previously unknown. It is noted that it contains all the classical vacuum Kerr–Schild spacetimes, excepting the plane-fronted gravitational waves, and hence the Kerr metric [1] is included too. It is also pointed out that the generalization of the Kerr metric in the Einstein background, due to Vaidya [16], is also included. However, the family of stationary spacetimes found is much more general and depends on an arbitrary function of one variable. On the other hand, the efficiency of the procedure is shown in section VI, where the Killing equations are completely solved without using coordinates. It is remarkable that, in general, these spacetimes have only one symmetry. Moreover, we discuss the conditions in which other symmetries can appear and their consequences on the GKS spacetimes.

The main equations in this paper are written in the Newman–Penrose formalism [17]. The conventions that we use here are the same of [5] with the only exception that the name for the main vector field of the null basis here is $\vec{\ell}$ whereas in [5] it is \vec{k} . Moreover, throughout this paper we have used units in which $8\pi G = c = 1$. Latin indexes run from 0 to 3. The abbreviation “c.c.” will stand for complex conjugate.

II Basic ingredients for the GKS transformation

In the classical Kerr–Schild transformation the seed spacetime was the simplest one, the Minkowski spacetime, being the important object the null vector field of the transformation, which was the most general SFGN vector field for Minkowski spacetime. Following this line, we take as seeds the subclass of the FLRW spacetimes with constant scale factor and their most general SFGN vector field as the vector field $\vec{\ell}$ for the transformation.

In what follows, we give the characterization and the explicit form of the seed spacetimes g_{ab} as well as of the SFGN vector fields $\vec{\ell}$.

The seed metrics

As we have said, the seed spacetimes are the FLRW spacetimes with constant scale factor, or equivalently, without expansion. Taking into account the form of the FLRW spacetimes given in appendix A, the line element for the seed spacetimes can be written in the following explicitly conformally-flat form

$$ds^2 = \frac{a^2}{(1 + \varepsilon U^2)(1 + \varepsilon V^2)} \left\{ -4 dU dV + 4 \left(\frac{V - U}{1 + \xi \bar{\xi}} \right)^2 d\xi d\bar{\xi} \right\}, \quad (2)$$

where a is the constant scale factor and a bar denotes complex conjugation. In the case $\varepsilon = 0$ this line element corresponds to the Minkowski spacetime in double null coordinates; in the case $\varepsilon = 1$ to the Einstein static Universe and, in the case $\varepsilon = -1$, it corresponds to a spacetime usually not considered because the energy density is negative, however we consider it as a seed spacetime because it may lead to GKS solutions with good physical properties.

These conformally-flat spacetimes have a perfect-fluid matter content and therefore, its energy-momentum tensor is given by

$$T_{ab} = (\varrho + p) u_a u_b + p g_{ab}, \quad g_{ab} u^a u^b = -1,$$

where the energy density ϱ and the pressure p are constant

$$\varrho = \frac{3\varepsilon}{a^2} = -3p,$$

and in addition, the fluid velocity is

$$\vec{u} = u^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} = \frac{1}{2a} \left\{ (1 + \varepsilon U^2) \frac{\partial}{\partial U} + (1 + \varepsilon V^2) \frac{\partial}{\partial V} \right\},$$

$$\mathbf{u} = u_a dx^a = -dt = -a \left\{ \frac{dU}{1 + \varepsilon U^2} + \frac{dV}{1 + \varepsilon V^2} \right\}.$$

It can be checked that \vec{u} is a constant vector field, that is $\nabla_a u_b = 0$. On the other hand, the case $\varepsilon = 0$ is Minkowski ($\varrho = p = 0$), and although we can define \vec{u} , it is not a preferred vector field.

The shear-free geodesic null vector field $\vec{\ell}$.

In analogy with the pioneering works [4] on the classical Kerr-Schild transformation, we restrict ourselves to the class of shear-free geodesic null vector fields. Although the

Goldberg–Sachs theorem [18] tells us that such a kind of vector fields may not exist in general spacetimes, it is common knowledge that in the Minkowski spacetime there is a big family of such vector fields and all of them are known explicitly: this result constitutes the so-called *Kerr’s theorem* (see [5, 19]). Moreover, it is well known that a SFGN vector field for Minkowski is also a SFGN vector field for any conformally–flat spacetime, which means that we have the same large class of such vector fields for the seed spacetimes (2). Here, we are going to construct all of them and after this, we will construct an appropriate null basis associated with them.

To begin with, the most general null one–form field for the seed spacetimes (2) can be written in the following form

$$\ell = F \left\{ dU + Y \bar{Y} dV + \frac{V - U}{1 + \xi \bar{\xi}} (\bar{Y} d\xi + Y d\bar{\xi}) \right\}, \quad (3)$$

where F and Y are real and complex arbitrary scalar fields respectively. Then, in order that ℓ be geodesic and shear–free, the complex function Y must be a solution of the following system of non–linear partial differential equations

$$(1 + \xi \bar{\xi}) Y Y_{,\xi} = (V - U) Y_{,V} + Y (1 + \bar{\xi} Y), \quad (4)$$

$$(1 + \xi \bar{\xi}) Y_{,\bar{\xi}} = (V - U) Y Y_{,U} - Y (Y - \xi), \quad (5)$$

where commas stand for partial derivative with respect to the subscript that follows.

Now, in order to simplify further calculations, we take a null basis $\{\ell, k, m, \bar{m}\}$ for the seed spacetimes (2) associated with $\vec{\ell}$. We choose this null basis in such a way that $\vec{\ell}$ be affinely parametrized and such that the unit timelike vector field \vec{u} can be written in the following way

$$\vec{u} = \frac{1}{\sqrt{2}} (\vec{\ell} + \vec{k}), \quad \mathbf{u} = \frac{1}{\sqrt{2}} (\ell + k).$$

Part of the remaining freedom is used for setting the imaginary part of the spin coefficient ϵ equals to zero. After some calculations, the explicit expressions for a such null basis $\{\ell, k, m, \bar{m}\}$ are (3) and

$$\begin{aligned} k = F \left\{ \frac{1 + \varepsilon V^2}{1 + \varepsilon U^2} Y \bar{Y} dU + \frac{1 + \varepsilon U^2}{1 + \varepsilon V^2} dV - \frac{V - U}{1 + \xi \bar{\xi}} (\bar{Y} d\xi + Y d\bar{\xi}) \right\}, \\ m = - F \left\{ \sqrt{Y \bar{Y}} \left(-\sqrt{\frac{1 + \varepsilon V^2}{1 + \varepsilon U^2}} dU + \sqrt{\frac{1 + \varepsilon U^2}{1 + \varepsilon V^2}} dV \right) + \right. \\ \left. \frac{V - U}{1 + \xi \bar{\xi}} \sqrt{\frac{\bar{Y}}{Y}} \left(\sqrt{\frac{1 + \varepsilon U^2}{1 + \varepsilon V^2}} d\xi - Y^2 \sqrt{\frac{1 + \varepsilon V^2}{1 + \varepsilon U^2}} d\bar{\xi} \right) \right\}, \end{aligned}$$

where the case $Y = 0$ can be obtained by taking $Y/\bar{Y} = 1$, and F is the following function

$$F = \frac{-\sqrt{2}a}{(1 + \varepsilon U^2) + (1 + \varepsilon V^2)Y\bar{Y}}.$$

After long but straightforward calculations we find that the spin coefficients associated with this null basis satisfy the relations

$$\begin{aligned} \kappa &= \sigma = \epsilon = \pi = \lambda = 0, \\ \tau - \bar{\nu} &= \rho - \bar{\mu} = \alpha + \bar{\beta} = \gamma + \bar{\gamma} = 0. \end{aligned} \quad (6)$$

Furthermore, the Newman–Penrose symbols for the Riemann tensor are

Ricci tensor:

$$\Phi_{01} = \Phi_{02} = \Phi_{12} = 0, \quad \Phi_{00} = 2\Phi_{11} = \Phi_{22} = \frac{1}{4}(\varrho + p) = \frac{\varepsilon}{2a^2}, \quad \Lambda = \frac{1}{24}(\varrho - 3p) = \frac{\varepsilon}{4a^2},$$

Weyl tensor:

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0.$$

III The construction of the GKS spacetimes

In this section we study the GKS transformation (1) for the seed spacetimes (2) with (3) as the SFGN vector field $\vec{\ell}$. To that end, we use the fact that given a null basis $\{\ell, k, m, \bar{m}\}$ associated with the seed metric, we can construct a null basis $\{\tilde{\ell}, \tilde{k}, \tilde{m}, \tilde{\bar{m}}\}$ associated with the GKS metric by taking (see [12, 6])

$$\begin{aligned} \tilde{\ell}^a &= \ell^a, & \tilde{\ell}_a &= \ell_a, \\ \tilde{k}^a &= k^a + \mathcal{H}\ell^a, & \tilde{k}_a &= k_a - \mathcal{H}\ell_a, \\ \tilde{m}^a &= m^a, & \tilde{m}_a &= m_a, \end{aligned} \quad (7)$$

where, from now on, objects with tilde are associated with the GKS spacetimes in contrast with objects without tilde which are associated with the seed spacetimes.

These relationships between null bases allow us to compute all objects associated with the GKS spacetimes as functions of the same objects associated with the seed spacetimes, and the function \mathcal{H} and their derivatives. In this sense, we can compute all the Newman–Penrose symbols for the GKS spacetimes. Taking into account the expressions of the previous section, the result for the components of the Ricci tensor is the following

$$\tilde{\Phi}_{00} = \frac{\varepsilon}{2a^2}, \quad \tilde{\Phi}_{01} = \tilde{\Phi}_{02} = 0, \quad (8)$$

$$\tilde{\Phi}_{11} = \frac{\varepsilon}{4a^2} + \frac{1}{2} \left[(\rho - \bar{\rho})^2 + \frac{\varepsilon}{2a^2} \right] \mathcal{H} + \frac{1}{4}(\rho + \bar{\rho}) \mathcal{V} - \frac{1}{4} D \mathcal{V}, \quad (9)$$

$$\tilde{\Phi}_{12} = [\delta \bar{\rho} + (\rho - \bar{\rho}) \tau] \mathcal{H} + \bar{\rho} \mathcal{H} + \frac{1}{2} (\tau \mathcal{V} - \delta \mathcal{V}), \quad (10)$$

$$\begin{aligned} \tilde{\Phi}_{22} = & \frac{\varepsilon}{2a^2} + \frac{1}{2} \left[2 \Delta(\rho + \bar{\rho}) - (\rho - \bar{\rho})^2 \right] \mathcal{H} + \frac{1}{2}(\rho + \bar{\rho}) \Delta \mathcal{H} + \frac{1}{2}(\rho + \bar{\rho}) \mathcal{V} + \\ & (\bar{\tau} + \alpha) \mathcal{U} + (\tau + \bar{\alpha}) \bar{\mathcal{U}} - \frac{1}{2} (\delta \bar{\mathcal{U}} + \bar{\delta} \mathcal{U}) + \frac{\varepsilon}{2a^2} \mathcal{H}^2, \end{aligned} \quad (11)$$

$$\tilde{\Lambda} = \frac{\varepsilon}{4a^2} + \frac{1}{6} \left[(\rho + \bar{\rho})^2 + 2\rho\bar{\rho} - \frac{3\varepsilon}{2a^2} \right] \mathcal{H} - \frac{1}{4}(\rho + \bar{\rho}) \mathcal{V} + \frac{1}{12} D \mathcal{V}, \quad (12)$$

where \mathcal{V} and \mathcal{U} are defined by

$$\mathcal{V} \equiv D \mathcal{H} + (\rho + \bar{\rho}) \mathcal{H}, \quad \mathcal{U} \equiv \delta \mathcal{H}.$$

In the same way, we can obtain the components of the Weyl tensor. Their expressions are

$$\tilde{\Psi}_0 = 0, \quad \tilde{\Psi}_1 = 0, \quad (13)$$

$$3 \tilde{\Psi}_2 = -(\rho - \bar{\rho})^2 \mathcal{H} - \frac{3}{2}(\rho - \bar{\rho}) \mathcal{V} - \frac{1}{2} D \mathcal{V}, \quad (14)$$

$$\tilde{\Psi}_3 = -(\rho - \bar{\rho}) \bar{\mathcal{U}} + \frac{1}{2}(\bar{\tau} \mathcal{V} - \bar{\delta} \mathcal{V}), \quad \tilde{\Psi}_4 = 2(\bar{\tau} - \alpha) \bar{\mathcal{U}} - \bar{\delta} \bar{\mathcal{U}}. \quad (15)$$

As we can see directly from these equations, all the GKS spacetimes that we can obtain are algebraically special because the null vector field $\vec{\ell} = \vec{\ell}$ is a multiple null eigenvector of the Weyl tensor.

Now, we must study the Einstein field equations for the GKS metrics with a perfect-fluid source. They read as follows

$$\tilde{G}_{ab} + C \tilde{g}_{ab} = \tilde{T}_{ab}, \quad (16)$$

where \tilde{G}_{ab} is the Einstein tensor for the GKS metrics, C is the cosmological constant, and \tilde{T}_{ab} is the energy-momentum tensor, which has the form

$$\tilde{T}_{ab} = (\tilde{\rho} + \tilde{p}) \tilde{u}_a \tilde{u}_b + \tilde{p} \tilde{g}_{ab}, \quad \tilde{g}_{ab} \tilde{u}^a \tilde{u}^b = -1, \quad (17)$$

where $\tilde{\rho}$, \tilde{p} and \tilde{u} are the energy density, the pressure and the fluid velocity of the GKS perfect fluid, respectively. In this situation, we project the Einstein equations (16) onto the null basis $\{\tilde{\ell}, \tilde{k}, \tilde{m}, \tilde{\bar{m}}\}$ and then, using (17) we obtain other expressions for the Ricci tensor components

$$\tilde{\Phi}_{00} = \frac{1}{2} (\tilde{\rho} + \tilde{p}) (\tilde{\ell}^a \tilde{u}_a)^2, \quad (18)$$

$$\tilde{\Phi}_{01} = \frac{1}{2} (\tilde{\varrho} + \tilde{p}) (\tilde{\ell}^a \tilde{u}_a) (\tilde{m}^b \tilde{u}_b), \quad \tilde{\Phi}_{02} = \frac{1}{2} (\tilde{\varrho} + \tilde{p}) (\tilde{m}^a \tilde{u}_a)^2, \quad (19)$$

$$\tilde{\Phi}_{11} = \frac{1}{4} (\tilde{\varrho} + \tilde{p}) \left[(\tilde{\ell}^a \tilde{u}_a) (\tilde{k}^b \tilde{u}_b) + (\tilde{m}^a \tilde{u}_a) (\tilde{m}^b \tilde{u}_b) \right], \quad (20)$$

$$\tilde{\Phi}_{12} = \frac{1}{2} (\tilde{\varrho} + \tilde{p}) (\tilde{k}^a \tilde{u}_a) (\tilde{m}^a \tilde{u}_a), \quad \tilde{\Phi}_{22} = \frac{1}{2} (\tilde{\varrho} + \tilde{p}) (\tilde{k}^a \tilde{u}_a)^2, \quad (21)$$

$$\tilde{\Lambda} = \frac{1}{24} (\tilde{\varrho} - 3\tilde{p}) + \frac{1}{6} C. \quad (22)$$

The next step in this process is to compare the expressions for the Ricci tensor (8–12) with the Einstein equations for the GKS metrics (18–22). We do this for the seed metrics with $\varepsilon \neq 0$, so that $\tilde{\varrho} + \tilde{p} \neq 0$, and later we will extend the results for the $\varepsilon = 0$ case. The outcome of this comparison is

- The components, in the null bases $\{\ell, k, m, \bar{m}\}$ and $\{\tilde{\ell}, \tilde{k}, \tilde{m}, \tilde{\bar{m}}\}$, for the fluid velocity \vec{u} of the GKS perfect fluid:

$$(\tilde{\ell}^a \tilde{u}_a)^2 = (\ell^a \tilde{u}_a)^2 = \frac{\varepsilon}{2a^2} \left[\frac{\varepsilon}{a^2} + 2 \left((\rho - \bar{\rho})^2 + \frac{\varepsilon}{2a^2} \right) \mathcal{H} + (\rho + \bar{\rho}) \mathcal{V} - D \mathcal{V} \right]^{-1}, \quad (23)$$

$$\tilde{k}^a \tilde{u}_a = k^a \tilde{u}_a + \mathcal{H} \ell^a \tilde{u}_a = \left[2 (\tilde{\ell}^a \tilde{u}_a) \right]^{-1}, \quad (24)$$

$$\tilde{m}^a \tilde{u}_a = m^a \tilde{u}_a = 0. \quad (25)$$

- The energy density and the pressure of the GKS perfect fluid:

$$\tilde{\varrho} = \frac{3\varepsilon}{a^2} - C + 2(2\rho^2 - \rho\bar{\rho} + 2\bar{\rho}^2) \mathcal{H} - D \mathcal{V}, \quad (26)$$

$$\tilde{p} = -\frac{\varepsilon}{a^2} + C + 2 \left(-3\rho\bar{\rho} + \frac{\varepsilon}{a^2} \right) \mathcal{H} + 2(\rho + \bar{\rho}) \mathcal{V} - D \mathcal{V}, \quad (27)$$

- Two second order partial differential equations for \mathcal{H} :

$$\delta \mathcal{V} = \tau \mathcal{V} + 2\bar{\rho} \mathcal{U} + 2[\delta \bar{\rho} + (\rho - \bar{\rho}) \tau] \mathcal{H}, \quad (28)$$

$$\begin{aligned} (\rho + \bar{\rho}) \triangle \mathcal{H} &= \left[(\rho - \bar{\rho})^2 + \frac{\varepsilon}{a^2} - 2 \triangle (\rho + \bar{\rho}) \right] \mathcal{H} - (\rho + \bar{\rho}) \mathcal{V} + \\ &\delta \bar{\mathcal{U}} + \bar{\delta} \mathcal{U} - 2(\bar{\tau} + \alpha) \mathcal{U} - 2(\tau + \bar{\alpha}) \bar{\mathcal{U}} + \left(\frac{\varepsilon}{a^2} \right)^{-1} \mathcal{W}, \end{aligned} \quad (29)$$

where \mathcal{W} is given by

$$\begin{aligned} \mathcal{W} &= \left[D \mathcal{V} - (\rho + \bar{\rho}) \mathcal{V} - 2(\rho - \bar{\rho})^2 \mathcal{H} \right] \times \\ &\left[D \mathcal{V} - (\rho + \bar{\rho}) \mathcal{V} - 2(\rho - \bar{\rho})^2 \mathcal{H} - \frac{2\varepsilon}{a^2} (1 + \mathcal{H}) \right]. \end{aligned} \quad (30)$$

As we can see, equation (29) is a non-linear partial differential equation for \mathcal{H} due to the term with \mathcal{W} .

IV Characterization of the GKS spacetimes through the kinematical quantities

Until now, the only assumption we have made on the GKS spacetimes is that their energy–momentum tensor must be of the perfect–fluid type. Then, once we have a function \mathcal{H} solution of the system (28–29), we have the explicit form of the GKS metric as well as of the matter content variables. However, there is not a systematic way of solving (28–29), specially because (29) is a non–linear equation. For this reason, we are going to introduce some additional assumptions on the GKS spacetimes which can help us to integrate these equations. Of course, it would be advisable to impose conditions which have a physical meaning. In this sense, an interesting way to control these conditions is through the study of the kinematical quantities for the fluid velocity \vec{u} of the GKS perfect fluid.

From equations (23–25) it follows that \vec{u} lies in the two–planes generated by $\vec{\ell}$ and \vec{k} . Then, by making the following change of basis

$$\vec{\ell} \rightarrow \vec{L} = \mathcal{A} \vec{\ell}, \quad \vec{k} \rightarrow \vec{K} = \mathcal{A}^{-1} \vec{k}, \quad \vec{m} \rightarrow \vec{\tilde{m}}, \quad (31)$$

where

$$\mathcal{A}^2 \equiv \left[2 (\ell^a \tilde{u}_a)^2 \right]^{-1}, \quad (32)$$

we get the following form for \vec{u}

$$\vec{u} = \frac{1}{\sqrt{2}} (\vec{L} + \vec{K}).$$

Therefore, we can now use the formulas given in the appendix B to obtain the kinematical quantities of \vec{u} . To that end, we compute the spin coefficients associated with the null basis $\{L, K, \tilde{m}, \tilde{\tilde{m}}\}$, which we denote with a hat. The most useful results are

$$\begin{aligned} \hat{\rho} + \hat{\bar{\rho}} &= (\rho + \bar{\rho}) \mathcal{A}, & \hat{\mu} + \hat{\bar{\mu}} &= (\rho + \bar{\rho})(1 - \mathcal{H}) \mathcal{A}^{-1}, & \hat{\epsilon} + \hat{\bar{\epsilon}} &= D\mathcal{A}, \\ \hat{\gamma} + \hat{\bar{\gamma}} &= -\mathcal{A}^{-1} D\mathcal{H} + (\Delta\mathcal{A} + \mathcal{H} D\mathcal{A}) \mathcal{A}^{-2}, & \hat{\alpha} + \hat{\bar{\beta}} &= \mathcal{A}^{-1} \bar{\delta}\mathcal{A}, \\ \hat{\pi} + \hat{\bar{\tau}} - \hat{\nu} - \hat{\bar{\kappa}} &= \bar{\tau} + [\bar{\delta}\mathcal{H} - \bar{\tau}(1 + \mathcal{H})] \mathcal{A}^{-2}, & \hat{\sigma} = \hat{\lambda} &= 0, \\ \hat{\rho} - \hat{\bar{\rho}} &= (\rho - \bar{\rho}) \mathcal{A}, & \hat{\mu} - \hat{\bar{\mu}} &= -(\rho - \bar{\rho})(1 + \mathcal{H}) \mathcal{A}^{-1}, \end{aligned}$$

In this situation, we are going to impose conditions which can lead to stationary spacetimes. To that end, we impose the vanishing of the shear and expansion of the fluid velocity \vec{u} , that is,

$$\tilde{\theta} = \tilde{\sigma}_{ab} = 0. \quad (33)$$

These conditions are necessary in order to have \vec{u} proportional to a time-like Killing vector field. Then, from the vanishing of the expansion and the $m_{(a}\bar{m}_{b)}$ -component of the shear tensor (see appendix B) we have

$$(\rho + \bar{\rho}) (\mathcal{A}^2 - 1 + \mathcal{H}) = 0,$$

so that, two different cases may appear: $\rho + \bar{\rho} = 0$ or $\mathcal{A}^2 = 1 - \mathcal{H}$. If $\rho + \bar{\rho} = 0$ the SFGN vector field $\vec{\ell}$ is expansion-free. In addition, from the Newman-Penrose equations [17] we obtain that $\rho^2 + \varepsilon/(2a^2) = 0$, which means that only the case $\varepsilon = 1$ is possible. However, further analysis of the equations for this case shows that there are not solutions in this case. Therefore, from now on we will consider that $\rho + \bar{\rho} \neq 0$, and then we must follow through the second possibility

$$\mathcal{A}^2 = 1 - \mathcal{H}. \quad (34)$$

From this equation and using the expressions (23, 32) we get another differential equation for \mathcal{H}

$$D\mathcal{V} = (\rho + \bar{\rho})\mathcal{V} + 2 \left[(\rho - \bar{\rho})^2 + \frac{\varepsilon}{a^2} \right] \mathcal{H}. \quad (35)$$

If we use this equation, we can see from (30) that $\mathcal{W} = -4(\varepsilon^2/a^2)\mathcal{H}$, and therefore (29) is a linear differential equation for \mathcal{H} . Moreover, from the remaining components of equations (33) we have the following two additional conditions on \mathcal{H} and Y ,

$$(D + \Delta)\mathcal{H} = 0, \quad (36)$$

$$\tau = 0, \quad (37)$$

respectively. The first is a further differential equation for \mathcal{H} , while the second one is a constraint on the possible complex functions Y that we can use to construct the SFGN vector fields $\vec{\ell}$. Then, replacing (34) into expressions (23–25) for the fluid velocity \vec{u} we obtain

$$\vec{u} = \frac{1}{\sqrt{2(1-\mathcal{H})}} \left[(1-\mathcal{H})\vec{\ell} + \vec{k} \right] = \frac{1}{\sqrt{2(1-\mathcal{H})}} (\vec{\ell} + \vec{k}) = \frac{1}{\sqrt{1-\mathcal{H}}} \vec{u}, \quad (38)$$

that is to say, \vec{u} is proportional to \vec{u} . On the other hand, it can be shown that the vector field \vec{u} , which is a Killing vector field for the seed spacetimes (2), is also a Killing vector field for the GKS spacetimes provided that (36) and (37) hold. Thus, \vec{u} is parallel to a Killing vector field. We can see this fact from the following expression

$$\tilde{\nabla}_a \tilde{u}'_b + \tilde{\nabla}_b \tilde{u}'_a = \sqrt{2} \left\{ (D + \Delta)\mathcal{H} \ell_a \ell_b + 2 \left[\tau k_{(a} m_{b)} + \text{c.c.} \right] \mathcal{H} \right\} = 0, \quad (39)$$

where $\tilde{u}'_a = \tilde{g}_{ab} u^b$. Here, we have used (6) and the second equality follows from (36,37). Notice that this is not a trivial result because, even though conditions (33) are necessary for \vec{u} to be proportional to a Killing, they are not sufficient (we also need the acceleration

$\tilde{\mathbf{a}}$ to be an exact 1-form). Therefore, we have proved that (33) implies $\mathbf{d}\tilde{\mathbf{a}} = \mathbf{0}$ under our conditions.

On the other hand, in the last section we restricted ourselves to the cases $\varepsilon = \pm 1$. Now, we can include the case $\varepsilon = 0$ by defining the vector field \vec{u} through the expression (38), and by taking the same partial differential equations (28–29) and (35–36) for \mathcal{H} .

The next point is to find which are the integrability conditions for the system of linear partial differential equations for \mathcal{H} . In this process we find two different cases.

- Case A: $\bar{\delta}\rho = 0 \Rightarrow \rho - \bar{\rho} = 0$. It can be shown that all the GKS metrics that we can find in this case are included in a particular case of reference [8], where the authors use the GKS transformation with the interior Schwarzschild metric as the seed metric. The resultant GKS spacetimes in that paper are stationary spherically symmetric and they contain a subfamily of regular solutions with equation of state $\tilde{\rho} + 3\tilde{p} = \text{const.}$. Among these spacetimes we have the Whittaker metric [20] as a special case, which, at the same time, is the static limit of the Wahlquist metric [21].

- Case B: $\bar{\delta}\rho \neq 0$. In this case the procedure to find the integrability conditions of the system of partial differential equations (28–29) and (35–36) for \mathcal{H} , leads us to the following equations for the first derivatives of \mathcal{H}

$$D\mathcal{H} = \frac{\rho^2 + \bar{\rho}^2 + \frac{\varepsilon}{a^2}}{\rho + \bar{\rho}} \mathcal{H} = -\Delta \mathcal{H}, \quad \delta \mathcal{H} = \frac{\delta \bar{\rho}}{\rho + \bar{\rho}} \mathcal{H}. \quad (40)$$

These equations can be easily solved without the explicit use of coordinates and the solution is

$$\mathcal{H} = m(\rho + \bar{\rho}), \quad (41)$$

where m is an arbitrary constant.

The energy–momentum tensor of these GKS spacetimes can be obtained by substituting \mathcal{H} in the explicit form of the fluid velocity, the energy density and the pressure, given in (38) and (26–27), respectively. In particular, the energy density and the pressure are

$$\tilde{\rho} = \frac{3\varepsilon}{a^2} [1 - m(\rho + \bar{\rho})] - C, \quad \tilde{p} = \frac{\varepsilon}{a^2} [-1 + m(\rho + \bar{\rho})] + C, \quad (42)$$

so that we have the following equation of state

$$\tilde{\rho} + 3\tilde{p} = 2C.$$

The most interesting case from the physical point of view is the one given by $C \geq 0$ and $\varepsilon = 1$, for which there are spacetime regions where all the energy conditions are fulfilled (see [22]). In this case, from the form (38) of \vec{u} , it follows that the region of the spacetime in which we can define this unit timelike vector field is

$$1 - \mathcal{H} > 0.$$

Moreover, the case $\varepsilon = C = 0$ corresponds to vacuum solutions. In the next section we will analyze to which solutions correspond each case.

On the other hand, from the expressions (13–15) for the Weyl tensor we already know that the GKS spacetimes are algebraically special. In this case, the Petrov type is II in general, except when the following relation holds

$$3 \tilde{\Psi}_2 \tilde{\Psi}_4 = 2 \tilde{\Psi}_3^2 \implies (\rho^2 + \frac{\varepsilon}{2a^2})(\bar{\delta}\bar{\delta}\rho + 2\alpha\bar{\delta}\rho) = 3\rho(\bar{\delta}\rho)^2. \quad (43)$$

In that case the Petrov type is D and the fluid velocity does not lie in the preferred two-space spanned by the two multiple null eigenvectors of the Weyl tensor.

Finally, the non vanishing kinematical quantities for \vec{u} are

$$\begin{aligned} \vec{\tilde{a}} &= -\frac{D\mathcal{H}}{\sqrt{2(1-\mathcal{H})}}\vec{v} - \left(\frac{\bar{\delta}\mathcal{H}}{1-\mathcal{H}}\vec{\tilde{m}} + \text{c.c.}\right), \\ \tilde{\omega}_{ab} &= -\left(\frac{\bar{\delta}\mathcal{H}}{1-\mathcal{H}}\tilde{v}_{[a}\tilde{m}_{b]} + \text{c.c.}\right) - \frac{\sqrt{2}(\rho-\bar{\rho})\mathcal{H}}{\sqrt{1-\mathcal{H}}}\tilde{m}_{[a}\tilde{\tilde{m}}_{b]}, \end{aligned} \quad (44)$$

where \tilde{v}^a is a unit spacelike vector field, orthogonal to \tilde{u}^a , and defined as follows

$$\tilde{v}^a \equiv \frac{(1-\mathcal{H})\tilde{\ell}^a - \tilde{k}^a}{\sqrt{2(1-\mathcal{H})}} \implies \tilde{v}^a\tilde{v}_a = 1, \quad \tilde{v}^a\tilde{u}_a = 0.$$

It is interesting to note that from expression (44) and from equations (40–41) we can deduce that the rotation $\tilde{\omega}_{ab}$ of \tilde{u}^a vanishes if and only if so does the rotation $\rho - \bar{\rho}$ of ℓ^a .

V Explicit form of the solutions

After the integration of the equations for \mathcal{H} , we are going to find an explicit expression for the line element of the GKS spacetimes. For this we must find explicitly all the SFGN vector fields $\vec{\ell}$ satisfying the condition (37), that is, we must find the explicit solutions of the partial differential equations (4–5) together with (37). To do that, it is more convenient to consider a new function Ω instead of Y , which is defined by

$$Y(U, V, \xi, \bar{\xi}) \equiv \sqrt{\frac{1+\varepsilon U^2}{1+\varepsilon V^2}} \Omega(t, \chi, \xi, \bar{\xi}),$$

and also, it is better to change from the coordinates $\{U, V\}$ to the $\{t, \chi\}$ ones (see appendix A). Once we have performed these changes, the equations (4–5) and (37) read as follows

$$(1 + \xi\bar{\xi})\Omega\Omega_{,\xi} - \bar{\xi}\Omega^2 = \Sigma\Omega_{,\chi} + \Sigma'\Omega, \quad (45)$$

$$(1 + \xi\bar{\xi})\Omega_{,\bar{\xi}} + \xi\Omega = -\Sigma\Omega\Omega_{,\chi} + \Sigma'\Omega^2, \quad (46)$$

$$\Omega_{,t} = 0, \quad (47)$$

where $\Sigma(\varepsilon, \chi)$ is given in appendix A. The most general solution of these equations can be given implicitly by

$$G\left(\frac{e^{-\sqrt{-\varepsilon}\chi}\Omega - \xi}{1 + \bar{\xi}e^{-\sqrt{-\varepsilon}\chi}\Omega}, \frac{(1 - \xi\bar{\xi})\Sigma'\Omega - \xi + \bar{\xi}\Omega^2}{(1 + \xi\bar{\xi})\Sigma\Omega}\right) = 0, \quad (48)$$

where $G(z_1, z_2)$ is any analytic complex function of two complex variables $\{z_1, z_2\}$. It is interesting to note that, in the same way, we could have found the general solution of only equations (45) and (46), then we would have obtained all the possible SFGN vector fields for the seed metrics (2), what constitutes the analogous result of the Kerr theorem. However, for the sake of brevity we do not give here the explicit form of the general solution. From the definition of the spin coefficient ρ , and using (45–47) we have

$$\rho = \begin{cases} \frac{1}{\sqrt{2}a} \frac{\Omega_{,\chi}}{\Omega} & \text{for } \Omega \neq 0, \\ \frac{1}{\sqrt{2}a} \frac{\Sigma'}{\Sigma} & \text{for } \Omega = 0, \end{cases}$$

and then, the solution (41) for \mathcal{H} is

$$\mathcal{H} = \begin{cases} \frac{m}{\sqrt{2}a} [\ln(\Omega\bar{\Omega})]_{,\chi} & \text{for } \Omega \neq 0, \\ \frac{\sqrt{2}m}{a} \frac{\Sigma'}{\Sigma} & \text{for } \Omega = 0. \end{cases}$$

Now, making the substitution of the expressions (41) for the function \mathcal{H} and (38) for the SFGN 1-form ℓ into the definition of the GKS transformation (2), we find the following explicit form for the line element of the GKS metrics

$$ds^2 = -dt^2 + a^2 \left(d\chi^2 + \frac{4\Sigma^2 d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2} \right) + \frac{a^2 \mathcal{H}}{(1 + \Omega\bar{\Omega})^2} \left[(1 + \Omega\bar{\Omega}) \frac{dt}{a} - (1 - \Omega\bar{\Omega}) d\chi + \frac{2\Sigma}{1 + \xi\bar{\xi}} (\bar{\Omega} d\xi + \Omega d\bar{\xi}) \right]^2. \quad (49)$$

It is interesting to point out that this family of stationary solutions of Einstein's equations has an arbitrary function, namely Ω . Moreover, as far as we know, these metrics were previously unknown.

With regard to the complex function Ω , if the function $G(z_1, z_2)$ does not depend on z_2 we have

$$\Omega = e^{\sqrt{-\varepsilon}\chi} f(\xi, \bar{\xi}),$$

where f is a complex function that we can find by solving the system of differential equations (45–46). This form of Ω implies that

$$\mathcal{H} = \begin{cases} 0 & \text{if } \varepsilon = 0, 1, \\ \frac{\sqrt{2}m}{a} & \text{if } \varepsilon = -1, \end{cases}$$

and this means that in order to avoid $\mathcal{H} = 0$ for the cases $\varepsilon = 0, 1$ (which would mean that the GKS transformation is trivial), we must impose that

$$\frac{\partial G}{\partial z_2} \neq 0,$$

and therefore, by the implicit function theorem, we can write (48) as follows

$$\Upsilon \left(\frac{e^{-\sqrt{-\varepsilon}\chi} \Omega - \xi}{1 + \xi e^{-\sqrt{-\varepsilon}\chi} \Omega} \right) + \frac{(1 - \xi \bar{\xi}) \Sigma' \Omega - \xi + \bar{\xi} \Omega^2}{(1 + \xi \bar{\xi}) \Sigma \Omega} = 0,$$

where $\Upsilon(z)$ is any analytic complex function of one complex variable z . Moreover, as we can see directly from the last equation, or from equations (45–47) and the expression for the line-element (49), the transformation

$$\Omega \longrightarrow \Omega' = -\frac{1}{\Omega}, \quad \chi \longrightarrow \chi' = -\chi,$$

is a *gauge* transformation since it induces the change $\vec{\ell} \longrightarrow \vec{\ell}' = -\vec{\ell}$, and therefore both solutions of equations (45–47) Ω and Ω' determine the same GKS metric. Apart from this gauge, we have not been able to find any other. On the other hand, we can think, as Debney [23] did, that part of the complex functions Ω are gauges of the seed metric, or in other words, that the GKS metrics we obtain are the seed metrics. Looking at the expressions (13–15) for the Weyl tensor, this can only happen when $\rho + \bar{\rho} = 0 \Rightarrow \bar{\delta}\rho = 0$, which corresponds to case A). Therefore, the GKS metrics (49) are always different from the seed metrics.

As we said before, the family of solutions (49) is too wide in the sense that it depends on the arbitrary function Ω . Analyzing the form of the metric (see also [10]) the solutions contained in each case are:

- Case $\varepsilon = 0$: If we do not consider the cosmological constant ($C = 0$), the metrics (49) are all the classical vacuum Kerr–Schild metrics [4] with the exception of the plane-fronted gravitational waves [24, 23, 25]. This exception is due to the fact that this class of solutions corresponds to the case $\rho = 0$ (see [5]). The Schwarzschild spacetime is obtained when $Y = \Omega = 0$, and the Kerr spacetime when we choose Υ in the following way

$$\Upsilon = -\frac{1}{\sqrt{2}ic}, \tag{50}$$

where c is an arbitrary real constant. In this case, if we carry out the suitable changes (see [10]), we can identify the mass and the angular momentum as measured from infinity with $m/(4\sqrt{2}a)$ and $(mc)/(4\sqrt{2}a)$ respectively.

- Case $\varepsilon = 1$: All the spacetimes within this case were unknown with the exception of the case given by (50), corresponding to a metric due to Vaidya [16] (it is also a particular

case of the Wahlquist metric [21, 26]) which has been interpreted as the Kerr metric in the cosmological background of the Einstein Universe. In this sense, the metrics of this class can be interpreted as the vacuum Kerr–Schild configurations in the background of the Einstein Universe.

- Case $\varepsilon = -1$: These spacetimes were also unknown and they correspond to the Kerr–Schild configurations in the background of the seed spacetimes (2) with $\varepsilon = -1$. From the physical point of view these spacetimes have less interest since they do not satisfy the usual energy conditions [22].

VI Solution of the Killing equations and study of the symmetries

The subject of this section is the study of the symmetries of the GKS spacetimes (49). A remarkable point in this study is the fact that it is possible to integrate the Killing equations without the explicit use of coordinates, as it happened with the equations for \mathcal{H} . To carry out this integration, we consider the Killing equations for these metrics

$$\tilde{\nabla}_a \zeta_b + \tilde{\nabla}_b \zeta_a = 0, \quad (51)$$

where $\vec{\zeta}$ is a hypothetical Killing vector field for the metrics (49), and we project them onto the null basis $\{\ell, k, m, \bar{m}\}$ associated with the seed spacetimes, what allows us to use its spin coefficients and their properties (6).

After making the integration (see [10] for details), we find that in the general case, that is to say, without additional assumptions on the complex function Ω , there is only one independent Killing vector field. Obviously, this Killing vector field is the timelike one that we already know

$$\vec{\zeta} = \frac{1}{\sqrt{2}} \left[(1 - \mathcal{H}) \vec{\ell} + \vec{k} \right] = \frac{1}{\sqrt{2}} (\vec{\ell} + \vec{k}) = \vec{u} \propto \vec{\tilde{u}}. \quad (52)$$

Thus, the solution (49) is stationary in general with no further symmetries. The norm of this Killing vector field is $\mathcal{H} - 1$, which for the cases $\varepsilon \pm 1$ is always negative [otherwise we cannot define $\vec{\tilde{u}}$ (38)], but in the case $\varepsilon = 0$ there are not restrictions and then, the sign of the norm divides the spacetime into three regions: A) $1 - \mathcal{H} > 0$, where the Killing is timelike. B) $1 - \mathcal{H} < 0$, where the Killing is spacelike. C) $1 - \mathcal{H} = 0$, where the Killing is null. In fact, that is the stationary limit hypersurface (this can be checked easily for the Schwarzschild and Kerr spacetimes). This division according to the norm of the Killing vector field (52) is an interesting fact because these three regions are usually seen as different spacetimes (see [5]).

Apart from the general case, there are specific cases with additional symmetries. These cases appear when the quantity $\Gamma + \bar{\Gamma}$ vanishes, where Γ is a complex function defined by

$$\Gamma \equiv \frac{1}{(\bar{\delta}\rho)^2} \left\{ (\rho^2 + \frac{\varepsilon}{2a^2})(\bar{\delta}\bar{\delta}\rho + 2\alpha\bar{\delta}\rho) - 3\rho(\bar{\delta}\rho)^2 \right\}. \quad (53)$$

Now, two different cases appear depending on whether the complex function Γ vanishes or not.

Case (i): $\Gamma \neq 0$. After integrating the Killing equations (51) we find that there are two independent Killing vector fields, which we can take as ζ^a , given in (52), and ζ_1^a , given by

$$\vec{\zeta}_1 = \frac{1}{\sqrt{|\Gamma|}} \left\{ (1 + \mathcal{H})\vec{\ell} - \vec{k} - \frac{2(\bar{\rho}^2 + \frac{\varepsilon}{2a^2})}{\delta\bar{\rho}}\vec{m} - \frac{2(\rho^2 + \frac{\varepsilon}{2a^2})}{\bar{\delta}\rho}\vec{\bar{m}} \right\}. \quad (54)$$

It is clear that we can write the most general Killing vector field as follows

$$C_1 \vec{\zeta} + C_2 \vec{\zeta}_1, \quad (55)$$

where C_1 and C_2 are two arbitrary constants. It can be shown that this general Killing vector field commutes with the timelike one $\vec{\zeta}$ given in (52). Furthermore, we can study the possible existence of an axial Killing vector field. To that end we must check the regularity condition

$$\frac{\tilde{\nabla}^a \zeta^2 \tilde{\nabla}_a \zeta^2}{4 \zeta^2} \xrightarrow{\text{axis}} 1, \quad \zeta^2 \equiv \zeta^a \zeta_a. \quad (56)$$

For the general Killing vector field (55) this condition is not satisfied for any value of C_1 and C_2 and therefore, there is no axial symmetry in this case.

Case (ii): $\Gamma = 0$. In this case, equation (43) implies that the GKS spacetimes (49) are Petrov type D. When we integrate the Killing equations (51), we find that there are also two independent Killing vector fields, which can be taken as (52) and

$$\vec{\zeta}_2 = \frac{1}{(\rho^2 + \frac{\varepsilon}{2a^2})^{\frac{3}{2}} (\bar{\rho}^2 + \frac{\varepsilon}{2a^2})^{\frac{3}{2}}} \left\{ \bar{\delta}\rho \delta\bar{\rho} \left[(1 + \mathcal{H})\vec{\ell} - \vec{k} \right] - 2(\bar{\rho}^2 + \frac{\varepsilon}{2a^2})\bar{\delta}\rho\vec{m} - 2(\rho^2 + \frac{\varepsilon}{2a^2})\delta\bar{\rho}\vec{\bar{m}} \right\}. \quad (57)$$

In the same way that in the case (i), we can check that the general expression for a Killing vector field $C_1 \vec{\zeta} + C_2 \vec{\zeta}_2$, where C_1 and C_2 are two arbitrary constants, commutes with the timelike Killing vector field $\vec{\zeta}$. In addition, and with regard to the possibility of axial symmetry, it can be shown that we can only have a spacelike Killing vector field vanishing in a certain region of the spacetime when $C_1 = 0$, and then, this region is defined by $|\bar{\delta}\rho| = 0$. Moreover, it can be checked that if the following relation holds

$$(\rho - \bar{\rho})^{-2} (\rho^2 + \frac{\varepsilon}{2a^2}) (\bar{\rho}^2 + \frac{\varepsilon}{2a^2}) \Big|_{|\bar{\delta}\rho| \rightarrow 0} \longrightarrow \text{const.}, \quad (58)$$

and we choose the constant C_2 as follows

$$C_2 = \sqrt{\frac{(\rho^2 + \frac{\varepsilon}{2a^2})(\bar{\rho}^2 + \frac{\varepsilon}{2a^2})}{-8(\rho - \bar{\rho})^2}} \Big|_{|\bar{\rho}| \rightarrow 0},$$

the axis regularity condition (56) is satisfied for the Killing vector field $C_2 \vec{\zeta}_2$, and therefore, it is an axial Killing vector field. In that case, the GKS spacetimes would be stationary axially symmetric spacetimes in *rigid rotation* because \vec{u} is proportional to the timelike Killing vector field (52). For the Kerr and the Kerr–Vaidya metrics, which are given by (50) and $\varepsilon = 0, 1$ respectively, it can be checked that the condition (58) is fulfilled. In general, following [27], we can conclude that all the spacetimes in this case with axial symmetry are subcases of the Wahlquist family [21].

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A Conformally–flat form of the FLRW spacetimes

In this appendix we are going to give a coordinate change to pass from the following well-known form for the line element of the FLRW spacetimes

$$ds^2 = -dt^2 + a^2(t) \left\{ d\chi^2 + \Sigma^2(\varepsilon, \chi) (d\theta^2 + \sin^2 \theta d\varphi^2) \right\}, \quad (59)$$

where $\varepsilon^3 = \varepsilon$ and $\Sigma(\varepsilon, \chi)$ satisfies the differential equation

$$\Sigma'^2 + \varepsilon \Sigma^2 = 1 \iff \Sigma(\varepsilon, \chi) = \begin{cases} \sin \chi & \text{if } \varepsilon = 1, \\ \chi & \text{if } \varepsilon = 0, \\ \sinh \chi & \text{if } \varepsilon = -1, \end{cases}$$

to the following explicitly conformally–flat form

$$ds^2 = \Phi^2(T, R) \left\{ -dT^2 + dR^2 + R^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right\}.$$

This transformation can be achieved by using the following coordinate change

$$T = \frac{2 \Sigma(\varepsilon, \psi)}{\Sigma'(\varepsilon, \psi) + \Sigma'(\varepsilon, \chi)}, \quad R = \frac{2 \Sigma(\varepsilon, \chi)}{\Sigma'(\varepsilon, \psi) + \Sigma'(\varepsilon, \chi)},$$

where ψ is the so-called *parametric time*, defined by

$$\dot{\psi} = \frac{d\psi}{dt} \equiv \frac{1}{a}.$$

After making this coordinate change, the conformal factor Φ becomes

$$\Phi^2 = \frac{a^2}{\left[1 + \varepsilon \left(\frac{T-R}{2}\right)^2\right] \left[1 + \varepsilon \left(\frac{T+R}{2}\right)^2\right]}.$$

In these coordinates the fact that the scale factor a depends only on t reads as follows

$$a = a\left(\frac{T}{1 - \frac{\varepsilon}{4}(T^2 - R^2)}\right).$$

Another interesting form for the line element (59) is obtained by passing to the following null spherical coordinates,

$$U = \frac{T-R}{2}, \quad V = \frac{T+R}{2}, \quad \xi = e^{i\varphi} \tan\left(\frac{\theta}{2}\right),$$

then, we obtain

$$ds^2 = \Phi^2(U, V) \left\{ -4dUdV + 4\left(\frac{V-U}{1+\xi\bar{\xi}}\right)^2 d\xi d\bar{\xi} \right\}, \quad \Phi^2(U, V) = \frac{a^2}{(1+\varepsilon U^2)(1+\varepsilon V^2)},$$

where

$$a = a\left(\frac{V+U}{1-\varepsilon UV}\right).$$

B Kinematical quantities and the Newman–Penrose formalism

In this appendix we give some useful formulae due to Wainwright [28], for the kinematical quantities of a unit timelike vector field. Let \vec{u} be any unit timelike vector field and let $\{\ell, k, m, \bar{m}\}$ be a null basis such that we can write down \vec{u} in the following way

$$\vec{u} = \frac{1}{\sqrt{2}}(\vec{\ell} + \vec{k}).$$

Then, we can write the kinematical quantities associated with \vec{u} using only the null basis $\{\ell, k, m, \bar{m}\}$ and the spin coefficients associated with them as follows

Expansion

$$\theta = \frac{1}{\sqrt{2}} (\epsilon + \bar{\epsilon} - \gamma - \bar{\gamma} + \mu + \bar{\mu} - \rho - \bar{\rho}) ,$$

Acceleration

$$\vec{a} = \frac{1}{\sqrt{2}} (\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}) \vec{v} + \frac{1}{2} \{ (\pi - \bar{\tau} + \nu - \bar{\kappa}) \vec{m} + \text{c.c.} \} ,$$

Shear

$$\begin{aligned} \sigma_{ab} = & \frac{1}{3\sqrt{2}} \{ 2(\epsilon + \bar{\epsilon}) + \rho + \bar{\rho} - 2(\gamma + \bar{\gamma}) - \mu - \bar{\mu} \} (v_a v_b - m_{(a} \bar{m}_{b)}) + \\ & \frac{1}{2} \left\{ \left(2(\alpha + \bar{\beta}) + \pi + \bar{\tau} - \nu - \bar{\kappa} \right) v_{(a} m_{b)} + \text{c.c.} \right\} + \frac{1}{\sqrt{2}} \{ (\lambda - \bar{\sigma}) m_a m_b + \text{c.c.} \} , \end{aligned}$$

Rotation

$$\omega_{ab} = \frac{1}{2} \left\{ \left(2(\alpha + \bar{\beta}) + \nu + \bar{\kappa} - \pi - \bar{\tau} \right) v_{[a} m_{b]} + \text{c.c.} \right\} + \frac{1}{\sqrt{2}} (\rho - \bar{\rho} + \mu - \bar{\mu}) m_{[a} \bar{m}_{b]} ,$$

where

$$\vec{v} \equiv \frac{1}{\sqrt{2}} (\vec{\ell} - \vec{k}) \quad \implies \quad v^a v_a = 1 , \quad v^a u_a = 0 .$$

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